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THE INVERSE PROBLEM OF THE TWO-DIMENSIONAL THEORY OF ELASTICITY IN THE HYDRODYNAMIC FORMULATION*

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The problem of optimizing the form of thin solids in a perfect fluid is considered. Its solution is constructed using the hydrodynamic analogy noted earlier /1,2/ in the inverse problem of the theory of elasticity, and is utilized to compute the attached masses.

Let a system of n thin solids bounded by smooth convex contours $\Gamma_l\ (l = 1, 2, \ldots, n)$ move through a perfect fluid at a constant unit velocity along the X-axis of a system of Cartesian coordinates. The motion is potential, irrotational and the fluid is at rest at infinity. We denote by S the region occupied by the solids, $\Gamma = U\Gamma_l$ its boundary, Σ the region of fluid flow complementing S to complete the plane, ν the normal to Γ exterior to S, and by x_ν, y_ν its direction cosines.

We consider the variational problem of minimizing, by chosing the form of Γ, the integral functional, i.e. the attached mass /3,4/ m_x of the system in the x-direction

$$m_x = -\rho \int_\Gamma \varphi_1 \frac{\partial \varphi_1}{\partial \nu}\, d\Gamma = \rho \int_\Sigma \operatorname{grad}^2 \varphi_1 dxdy \tag{1}$$

under isoperimetric constraints.

Here $\varphi_1(x, y)$ is the fluid velocity potential, harmonic in Σ and decreasing at infinity, and ρ is the fluid density. Henceforth we shall put $\rho = 1$. The condition of fluid impermeability holds on Γ for $\varphi_1(x, y)$

$$d\varphi_1/d\nu = -x_\nu \tag{2}$$

and we specify the total area of the system

$$A = \int_S dxdy = \frac{1}{2} \int_\Gamma ydx - xdy \tag{3}$$

as the isoperimetric condition.

If there are no other constraints, the problem has a trivial solution, and the bodies degenerate into "needles" stretched parallel to the X-axis. We have for them $m_x = 0$, and Eqs. (3) are formally satisfied when the length of the needle becomes infinite, for any value of A. To eliminate this solution we must impose additional requirements on the bodies, e.g. specify their form at the points of incidence and departure of the flow, and vary only that part of the boundary near the middle cross section /5/. It has been shown that at this part the fluid velocity must have a constant value. In the case of flow past a polygon with the formation of a cavitational void of prescribed area, this agrees with the result obtained by Riabushinskii (see /6/). If on the other hand the boundary is completely unknown by definition, its total length can be specified although such a non-linear condition causes considerable difficulties when solving the problem.

Instead of a straightforward geometrical constraint eliminating the needles, we shall specify an attached mass m_y of the system in the direction of the Y-axis. Using the obvious notation we have

$$m_y = -\int_\Gamma \varphi_2 \frac{\partial \varphi_2}{\partial \nu}\, d\Gamma = \int_\Sigma \operatorname{grad}^2 \varphi_2 dxdy \tag{4}$$

$$\partial\varphi_2/\partial\nu = -y_\nu \tag{5}$$

Clearly, m_y and ξ_x, the latter representing the characteristic dimension of the system in the direction of the x-axis, are related to each other implicitly, and finite values of the attached mass have the corresponding finite sizes. The constraint (4) appears to be artificial, but in the specific versions of the optimal boundary given below a direct dependence of m_y in ξ_x will be established. Such a formulation is of some practical interest in designing low drag aerodynamic and hydrodynamic profiles /5/.

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We will solve the variational problem (1) with constraints (3), (4), by constructing the extended functional

$$J=\int_{\Sigma} f(x,y)\,dxdy+\beta\int_{S} dx\,dy, \quad f(x,y)=\mathrm{grad}^2\varphi_1+\alpha\,\mathrm{grad}^2\varphi_2 \tag{6}$$

where the constants α, β are Lagrange multipliers. The expression for the first variation δJ of the functional of type (6) with a moving boundary, is given in /7/. The expression reduces to two terms, the Euler equation satisfied identically in Σ, and the condition of optimality of the boundary

$$(\mathbf{v},\delta\mathbf{r})f-(\mathbf{T}_1,\mathbf{v})(\nabla\varphi_1,\delta\mathbf{r})-(\mathbf{T}_2,\mathbf{v})(\nabla\varphi_2,\delta\mathbf{r})+\beta(\mathbf{v},\delta\mathbf{r})=0, \tag{7}$$
$$\delta\mathbf{r}=(\delta x,\delta y),\ \nabla\varphi_i=(\varphi_{ix},\varphi_{iy}),\quad \mathbf{T}_i=\left(\frac{\partial f}{\partial\varphi_{ix}},\frac{\partial f}{\partial\varphi_{iy}}\right),\quad i=1,2$$

where $(\mathbf{a},\mathbf{b})$ is the scalar product of the vectors a and $\mathbf{b}$.

Condition (7) must hold for any small variations δx, δy of the points on the contour in Cartesian coordinates. Let us change to the variations $\delta\mathbf{v}$, δt in the local system of coordinates at the boundary required (t is the tangent to Γ at the given point, s is the length of the contour arc, and x_s, y_s are the direction cosines of the tangent)

$$\delta\mathrm{v}=x_\mathrm{v}\delta x+y_\mathrm{v}\delta y;\quad \delta t=x_s\delta x+y_s\delta y \tag{8}$$
$$x_\mathrm{v}=y_s,\quad x_s=-y_\mathrm{v}$$
$$x_\mathrm{v}^2+y_\mathrm{v}^2=1,\quad x_s^2+y_s^2=1$$

Then the vector condition (7) becomes equivalent to two scalar conditions

$$\frac{\partial\varphi_1}{\partial\mathrm{v}}\frac{\partial\varphi_1}{\partial s}+\alpha\frac{\partial\varphi_2}{\partial\mathrm{v}}\frac{\partial\varphi_2}{\partial s}=0,\quad f-2\left(\frac{\partial\varphi_1}{\partial\mathrm{v}}\right)^2-2\left(\frac{\partial\varphi_2}{\partial\mathrm{v}}\right)^2+\beta=0 \tag{9}$$

Taking into account the boundary conditions (2), (5), relations (8) and the invariance of the gradient under orthogonal transformations, conditions (9) become

$$x_\mathrm{v}\frac{\partial\varphi_1}{\partial s}+\alpha y_\mathrm{v}\frac{\partial\varphi_2}{\partial s}=0 \tag{10}$$
$$\left(\frac{\partial\varphi_1}{\partial s}\right)^2-x_\mathrm{v}^2+\alpha\left(\frac{\partial\varphi_2}{\partial s}\right)^2-\alpha y_\mathrm{v}^2+\beta=0$$

These conditions are satisfied identically if and only if φ_1, and φ_2 take the following values at the optimal boundary:

$$\varphi_1(x,y)=ax+c_l,\quad \varphi_2(x,y)=by+d_l;\quad x,\ y\in\Gamma_l,\quad l=1,2,\ldots,n \tag{11}$$

The constants a, b, c_l, d_l $(l=1,2,\ldots,n)$ are to be determined later. Substituting (11) into (10), we obtain

$$a=\alpha b,\ a^2-\alpha=\alpha b^2-1,\ \beta=\alpha b^2-1$$

from which it follows that

$$a=b,\ \alpha=1,\ \beta=b^2-1 \tag{12}$$

or

$$a=b^{-1},\ \alpha=b^{-2},\ \beta=0 \tag{13}$$

Let us consider the stream function $\Psi_2(x,y)$, conjugate to $\varphi_2(x,y)$. The boundary conditions for φ_2 and the conjugation conditions /4/

$$\frac{\partial\varphi_2}{\partial\mathrm{v}}=\frac{\partial\Psi_2}{\partial s},\quad \frac{\partial\varphi_2}{\partial s}=-\frac{\partial\Psi_2}{\partial\mathrm{v}} \tag{14}$$

imply that the following relations hold for $\Psi_2(x,y)$ at the boundary sought:

$$\Psi_2(x,y)=x+h_l,\quad \frac{\partial\Psi_2(x,y)}{\partial\mathrm{v}}=-bx_\mathrm{v}; \tag{15}$$
$$x,y\in\Gamma_l,\quad l=1,2,\ldots,n$$

and hence the relations

$$\chi(x,y)=h_l-ac_l,\quad \frac{\partial\chi(x,y)}{\partial\mathrm{v}}=(ab-1)x_\mathrm{v};\quad x,\ y\in\Gamma_l$$

for $\chi(x,y)\equiv\varphi_1(x,y)-a\Psi_2(x,y)$. This implies that $\varkappa(x,y)$ takes its constant value on every contour Γ_l. The function represents the real part of some function holomorphic in Σ and decreasing at infinity. The conditions of solvability of the modified Dirichlet problem /8/

imply that $\varkappa(x, y) \equiv 0$ in Σ; hence $h_l = c_l$ and $ab = 1$. This holds always for the set of parameters (13), and in the case of (12) it holds when $a = b = 1$. Then (12) is equivalent to (13). In both cases $\Psi_2(x, y)$ is proportional to $\varphi_1(x, y)$.

The constant b is found from m_y. Substituting the second relation of (11) into (4) we obtain

$$m_y = -b\int_\Gamma (y + d_l) y_\nu d\Gamma = b\int_\Gamma y x_s ds = bA$$

i.e. b is equal to the (given) coefficient μ_y /3/ of the mass attached to the system in the y-direction. Then the minimizing coefficient is $\mu_x = \mu_y^{-1}$ and $m_x = A^2 m_y^{-1}$. The result has a clear physical meaning. The greater m_y, the greater, generally speaking, the dimension ξ_x, and hence the smaller the characteristic transverse size of the system for the given surface area, and hence also the perturbation in the fluid, i.e., m_x. We establish directly that in the case of condition (11) the inertial coefficient $\mu_{xy} = 0$. Thus the coordinate axes coincide with the principal directions of the system and its steady motion in the fluid is in fact possible /3/.

The determination of the boundary reduces in practice to solving the inverse problem of potential theory in Σ relative to $\varphi_1(x, y)$, for which we have the Dirichlet condition (11) and the Neumann condition (2). The constants c_l $(l = 1, 2, \ldots, n)$ are found from the conditions of solvability /8/. The boundary found on the strength of this extra information will be optimal even when instead of bodies in a fluid we consider a system of conductors in a homogeneous electrostatic field directed along the X-axis, and the attached masses are replaced by the values of the polarization P_x, P_y [9]. With this interpretation (11) now becomes the boundary condition of the original problem /9/ and (2), (5) become the necessary conditions of optimality. Finally, the same conditions appear in the following version of the inverse problem of the theory of elasticity, whose solution we shall utilize.

Let the region Σ be filled with a homogeneous and isotropic, linearly elastic material with shear modulus G. A homogeneous field of tensile stresses with components $\sigma_x^\infty, \sigma_y^\infty$ is defined at infinity, and a constant pressure p at the boundaries of the holes Γ_l $(l = 1, 2, \ldots, n)$. It was shown in /10,11/ that the first invariant of the stress tensor has a constant value at the optimal boundary, furnishing a minimum to the local functional, representing a maximum over the region of the Mieses plasticity criterion /9/. The variational derivation of this condition for the potential deformation energy density integral is given in /12/. It can be shown that in this case $\vartheta(x, y) \equiv 0$ also in Σ, where $\vartheta(x, y)$ denotes the relative volume expansion /13/ corresponding to the perturbation field with displacement components $u(x, y), v(x, y)$, brought to a uniform state of stress by the holes. Here the conjugate conditions (14) hold for u and v, and this makes it possible, using (8), to reduce the boundary conditions on Γ of the direct problem written on the inclined areas /13/ in terms of the stress tensor components,

$$-(q_1 + p)x_\nu = \sigma_x x_\nu + \tau_{xy} y_\nu, \quad -(q_2 + p) y_\nu = \tau_{xy} x_\nu + \sigma_y y_\nu$$

to the form

$$u(x, y) = 2Gq_2 x + g_l, \quad \frac{\partial u(x, y)}{\partial \nu} = -2Gq_1 x_\nu; \quad x, y \in \Gamma_l, \quad l = 1, 2, \ldots, n \tag{16}$$

from which we find that $u(x, y) \equiv \varphi_2(x, y)$, $v(x, y) \equiv \Psi_2(x, y)$ when $(q_1 + p)\ (q_2 + p)^{-1} = b$.

The boundary in problem (16) is found /1,14/ by a conformal mapping of the standard region onto the region sought, using the analogy mentioned above to give examples of flows past the optimal system of bodies. The determination of m_x, m_y reduces to computing the surface area of S from the explicit value given for Γ. The form of the boundary depends on the geometry of the standard region and the values of the load parameters.

Thus for $n = 1$, an ellipse is given as the optimal shape in /1/, and we have for it the following well-known results [3]: $m_x = \pi C^2 b^2$, $m_y = \pi C^2 a^2$ (Ca, Cb are the axes of the ellipse and C is a scale multiplier). When $p = 1$, $q_1, q_2 = 0$, we have $\mu_x = \mu_y = 1$, i.e. a system with equal masses attached in the x and y directions. In the case of cyclic symmetry its configuration for various values of n is given essentially in /14/.

For two bodies on the X-axis the equations of the boundary of the (right) body have the form /1/ (ζ is a parameter)

$$x = x_0 + \frac{Cq_1}{q_1 + q_2}(\zeta - \lambda_1), \quad \lambda_1 \leqslant \zeta \leqslant 1, \quad 0 \leqslant \lambda_1 \leqslant 1 \tag{17}$$

$$x_0 = \frac{Cq_1}{q_1 + q_2}\lambda_1 + \frac{Cq_2}{q_1 + q_2}[\mathbf{K}(\lambda_1)(1 + D_2) - \mathbf{E}(\lambda_1)]$$

$$y = \frac{Cq_2}{q_1 + q_2}[\mathbf{E}(k) - E(\gamma, k) + D_2\mathbf{K}(k) - D_2 F(\gamma, k)]$$

$$k^2 = 1 - \lambda_1^2, \quad \gamma = \arcsin\left(\frac{1 - \zeta^2}{1 - \lambda_1^2}\right)^{1/2},$$

$$D_2 = -\frac{\mathbf{E}(k)}{\mathbf{K}(k)}, \quad \mu_x = \mu_y^{-1} = q_2 q_1^{-1}$$

(F, E are the elliptic and $\mathbf{K}, \mathbf{E}$ the total integrals of first and second kind, respectively). Computing the area /15/ we obtain

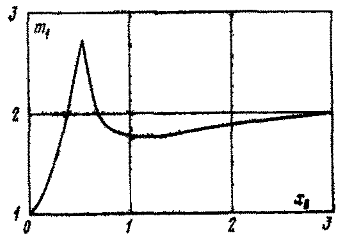
$$m_x = \frac{q_2}{q_1} A = 4\pi C^2 \frac{q_2}{q_1+q_2} [\lambda_1^2 + D_2 + 1] \tag{18}$$

$$m_y = \frac{q_1}{q_2} A = 4\pi C^2 \frac{q_1}{q_1+q_2} [\lambda_1^2 + D_2 + 1] \tag{19}$$

We find the relation connecting m_y with ξ_x just as simply. Taking as ξ_x e.g. the one of the bodies at the intersection with the X-axis

$$\xi_x = x(1) - x_0 = \frac{2Cq_1}{q_1+q_2}(1-\lambda_1) \tag{20}$$

we find that relations (19) and (20) yield the relation sought in terms of the parameters λ_1. The forms of the optimal contours for the case in question are given for various values of q_1/q_2 in /1/.



When $q_2=0$, the optimal bodies degenerate into plates of length $C(1-\lambda_1)$ on the X-axis, and (18) and (19) transform into the well-known result due to L.I. Sedov /3/. When $q_1 \ll 1$, we have bodies with small extensions for which computing m_x and m_y usually presents difficulties. When $q_1 = 0$, the bodies become plates perpendicular to the X-axis. As far as we know, the latter case has never been studied.

Assuming that the plates are of unit length, we obtain from (17) the following value for the scale factor:

$$C = Z^{-1}(\varphi, k), \quad \varphi = \arcsin\left(\frac{1-D_2^2}{1-\lambda_1^2}\right)^{1/2} \tag{21}$$

where $Z(\varphi, k)$ is the Jacobi Zeta function. The figure shows the relation between (17, (18), (21) $m_1 = m_x/\pi$ and x_0, the latter denoting the half distance between the plates.

Other configurations such as singly and doubly periodic networks of bodies, plates inclined at an arbitrary angle to the X-axis, etc., can be considered in the same manner.

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